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Noncommutative Schur functions and their applications

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Abstract

We develop a theory of Schur functions in noncommuting variables, assuming commutation relations that are satisfied in many well-known associative algebras. As an application of our theory, we prove Schur-positivity and obtain generalized Littlewood–Richardson and Murnaghan–Nakayama rules for a large class of symmetric functions, including stable Schubert and Grothendieck polynomials.

1. Introduction and main results

In this paper we develop a theory of Schur functions in noncommuting variables, assuming certain commutation relations that are satisfied in many well-known examples such as the plactic, nilplactic, and nilCoxeter algebras and the degenerate Hecke algebra $H_n(0)$. We show that most of the classical theory of symmetric functions can be reproduced in this noncommutative setting.

There are many combinatorial representations of these commutation relations, and to each of these one can associate a family of (ordinary) symmetric functions; examples of such families include skew Schur functions and stable Schubert and Grothendieck polynomials. As an application of our theory, we prove Schur-positivity of these functions and obtain generalized Littlewood–Richardson and Murnaghan–Nakayama rules for the corresponding characters of the symmetric group.

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Let $u_1,\ldots,u_n$ be elements of an associative algebra A. For a Ferrers shape λ, define the *(noncommutative) Schur function* $s_\lambda(\boldsymbol{u}) = s_\lambda(u_1,\ldots,u_n)$ by

$$s_\lambda(\boldsymbol{u}) = \sum_T \boldsymbol{u}^T.$$

In this formula, the sum ranges over all semi-standard tableaux T of shape λ and $\boldsymbol{u}^{\mathrm{T}}$ denotes the product $\prod u_i$, with indices i obtained by reading the columns of T bottom-up. For example, if $\lambda = (3,2)$, then there are two semi-standard tableaux

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&2\\\cline{1-2}\end{array}.$$

Thus we have

$$s_\lambda(u_1,u_2) = u_2u_1u_2u_1u_1 + u_2u_1u_2u_1u_2.$$

Similarly, for a skew shape λ/μ, one can define the (noncommutative) *skew* Schur function $s_{\lambda/\mu}(\boldsymbol{u}) = \sum_T \boldsymbol{u}^T$ where the sum is over all semi-standard skew tableaux T of shape λ/μ. Note that the $s_{\lambda/\mu}(\boldsymbol{u})$ are, in general, not symmetric in the u_i, and they will not be symmetric in most of our examples. When the u_i are commuting indeterminates, the above definition gives the ordinary Schur functions (see [24]).

The simplest instances of noncommutative Schur functions are the analogs of the elementary and complete homogeneous symmetric functions, given by

$$e_k(\boldsymbol{u}) = \sum_{a_1>a_2>\cdots>a_k} u_{a_1}u_{a_2}\cdots u_{a_k}$$

and

$$h_k(\boldsymbol{u}) = \sum_{a_1\leqslant a_2\leqslant\cdots\leqslant a_k} u_{a_1}u_{a_2}\cdots u_{a_k}.$$

Motivated by a large number of interesting examples (described in Sections 2 and 6), we will first try to extend the classical theory to cases where complete commutativity is replaced by 'nonlocal commutativity'

$$u_iu_k = u_ku_i, \quad |i-k|\geqslant 2; \tag{1.1}$$

or by the weaker 'non-local Knuth relations'

$$\begin{aligned} u_iu_ku_j &= u_ku_iu_j, \quad i\leqslant j<k,\ |i-k|\geqslant 2,\\ u_ju_iu_k &= u_ju_ku_i, \quad i<j\leqslant k,\ |i-k|\geqslant 2. \end{aligned} \tag{1.2}$$

A prominent example of this type was given by Schützenberger and Lascoux who constructed, in their pioneering papers [25,21], a theory of noncommutative Schur functions for the *plactic algebra* (see Example 2.1). We will show that important features of their theory can be extended to many other examples. In all of these examples, including the plactic algebra, there are additional 'local' commutation relations involving adjacent pairs of variables u_i and u_{i+1}. A single local relation that subsumes all of the examples we consider is the following:

$$(u_i + u_{i+1})u_{i+1}u_i = u_{i+1}u_i(u_i + u_{i+1}). \tag{1.3}$$

Note that (1.3) is equivalent to the statement that $e_1(u_i, u_{i+1})$ and $e_2(u_i, u_{i+1})$ commute.

Theorem 1.1. *Assume that $u_1, \ldots, u_n$ satisfy* (1.2) *and* (1.3). *Then the canonical map $s_{\lambda/\mu} \mapsto s_{\lambda/\mu}(\boldsymbol{u})$ extends to a homomorphism from the algebra Λ_n of ordinary symmetric functions in n commuting variables to the algebra $\Lambda_n(\boldsymbol{u})$ generated by the $s_\lambda(\boldsymbol{u})$. In particular,*

- *the $s_\lambda(\boldsymbol{u})$ commute;*
- *the $s_\lambda(\boldsymbol{u})$ span $\Lambda_n(\boldsymbol{u})$ as a $\mathbb{Z}$-module;*
- *the $s_\lambda(\boldsymbol{u})$ multiply according to the usual Littlewood–Richardson rule;*
- *the $s_{\lambda/\mu}(\boldsymbol{u})$ expand according to the usual Littlewood–Richardson rule.*

To make it more accessible, we restate the main message of Theorem 1.1 as follows:

> If non-adjacent variables commute (or satisfy the non–local Knuth relations) and adjacent variables $a < b$ satisfy
>
> $$aba + bba = baa + bab$$
>
> then noncommutative Schur functions behave as if they were ordinary Schur functions.

In particular, *whenever conditions* (1.2) *and* (1.3) *hold, all identities of the commutative theory are valid for the $s_\lambda(\boldsymbol{u})$*, as long as these identities can be stated solely in terms of Schur functions. In this way, one can obtain noncommutative versions of the Cauchy, Jacobi–Trudi, and Giambelli formulas, and many others.

Our main applications are obtained from what we call *combinatorial representations* of the commutation relations (1.2) and (1.3), in which the u_i are represented by *partial maps*. Let $\mathcal{P}$ be a finite or countable set, and let $\mathbb{R}\mathcal{P}$ be the vector space formally spanned by the elements of $\mathcal{P}$, with coefficients in $\mathbb{R}$. A representation of the variables u_i by linear operators in $\mathrm{End}(\mathbb{R}\mathcal{P})$ is called *combinatorial* if, for any u_i and for any $p \in \mathcal{P}$, the image $u_i p$ is either another element of $\mathcal{P}$ or zero. In other words, u_i can be viewed as a *partial map* $\mathcal{P} \to \mathcal{P}$ in which the image of an element is formally set to 0 whenever the map is not defined. The matrix representing u_i in the basis $\mathcal{P}$ is a partial permutation matrix, i.e., a 0-1 matrix each of whose columns contains at most one 1.

A typical example of such a representation is related to the *nilCoxeter algebra* of the symmetric group (see, e.g., [11]). Here $\mathcal{P}$ is the set of all permutations of $n + 1$

elements, and u_i acts by

$$u_i \cdot w = \begin{cases} \text{usual product } s_i w & \text{if } l(s_i w) = l(w) + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where s_i is the adjacent transposition $(i, i+1)$ and $l(w)$ denotes the length of a permutation w, i.e., the number of inversions. It is easy to see that the relations (1.1) and (1.3) are satisfied. This example and many others are discussed in more detail in Sections 2 and 6.

To any combinatorial representation of our relations one can associate a family $\{F_{h/g}\}$ of symmetric functions [7] which may be regarded as generalizations of skew Schur functions. Suppose that the u_i act as partial maps on a set $\mathscr{P}$, and let g and h be elements of $\mathscr{P}$. Define

$$F_{h/g}(x_1, \ldots, x_m) = \left\langle \prod_{i=1}^{m} \prod_{j=n}^{1} (1 + x_i u_j)\ g\ ,\ h \right\rangle,$$

where the x_i commute with each other and with the u_j, and the noncommuting factors of the double product are multiplied in the specified order. Here $\langle *, * \rangle$ denotes the inner product on $\mathbb{R}\mathscr{P}$ for which the elements of $\mathscr{P}$ form an orthonormal basis. Note that $F_{h/g}$ is a function in the commuting variables x_i alone; we shall later demonstrate that this is indeed a symmetric function, in the ordinary sense.

A purely combinatorial description of $F_{h/g}$ may be obtained by associating terms in the expansion of $\prod \prod (1 + x_i u_j)$ with sequences

$$g = g_0 \xrightarrow{u_{a_1}} g_1 \xrightarrow{u_{a_2}} \cdots \xrightarrow{u_{a_k}} g_k = h$$

and summing monomials in the x_i corresponding to each such sequence. When $g = \mu$ and $h = \lambda$ are Ferrers shapes, and the u_i are the Schur operators of [6] (i.e., u_i adds a box in column i), one obtains the standard definition of the skew Schur function $s_{\lambda/\mu}$.

Our formulation of the $F_{h/g}$ may also be expressed in the language of Gessel's quasi-symmetric functions [14], or Stanley's compatible sequences [2]. Other examples of symmetric functions constructed in this way have appeared in several places in the literature [26,10]. Note that $\prod \prod (1 + x_i u_j)$ can be viewed as a 'noncommutative Cauchy product' (cf. Sections 2 and 4).

It can be shown [7] that (1.2) and (1.3) imply that the functions $F_{h/g}$ are indeed symmetric, though of course this is far from obvious. In the case of the nilCoxeter algebra the above definition produces the *stable Schubert polynomials*, also known as Stanley symmetric functions [26]. Other examples are discussed in Section 2.

In this paper, we go further and prove Schur-positivity of the functions $F_{h/g}$ and obtain combinatorial rules for their expansions in terms of Schur functions.

Theorem 1.2 (Generalized Littlewood–Richardson rule). *Assume that $u_1, \ldots, u_n$ are partial maps $\mathscr{P} \to \mathscr{P}$ satisfying* (1.2) *and* (1.3). *Let g and h be elements of $\mathscr{P}$ and let*

$F_{h/g}(x_1,\dots,x_m)$ be defined as above. Then

- *the $F_{h/g}(x_1,\dots,x_m)$ are nonnegative integer combinations of ordinary Schur functions $s_\lambda(x_1,\dots,x_m)$;*
- *the coefficient $c^h_{g\lambda}$ of s_λ in the expansion of $F_{h/g}$ is given by*

$$c^h_{g\lambda} = \#\{T : |T| = \lambda' \ , \ w(T)g = h\}. \tag{1.4}$$

Here T is a semi-standard tableau, $|T|$ denotes the shape of T, and $w(T)$ denotes the column word of T, interpreted as a product of u_i's.

Using the classical construction of Frobenius, we can associate a representation of the symmetric group to each nonnegative integer combination of Schur functions. In particular, the functions $F_{h/g}$ correspond to certain representations which decompose into irreducibles with the above-described multiplicities. It would be very interesting to find natural (intrinsic) constructions of these representations; this problem has only been solved in some special cases.

A related natural problem is to compute the corresponding characters $\chi_{h/g}$, thus giving a generalization of the Murnaghan–Nakayama formula for an irreducible or, more generally, skew character of the symmetric group. A solution to this problem is given by the following theorem. Define w to be a *hook word* if $w = b_l b_{l-1} \cdots b_1 a_1 a_2 \cdots a_m$ where $b_l > b_{l-1} > \cdots > b_1 > a_1 \leqslant a_2 \leqslant \cdots \leqslant a_m$.

Theorem 1.3 (Generalized Murnaghan–Nakayama rule). *Assume the conditions of Theorem 1.2 hold. Then the value of the character $\chi_{h/g}$ on a conjugacy class $\alpha = (\alpha_1,\dots,\alpha_k)$ is given by*

$$\chi_{h/g}(\alpha) = \sum_{\substack{w_1,\dots,w_k \\ w_1\cdots w_k g = h}} (-1)^{\mathrm{asc}(w_1)+\cdots+\mathrm{asc}(w_k)} \tag{1.5}$$

where each w_i is a hook word of length α_i, and asc(w_i) *denotes the number of ascents in w_i.*

The paper is organized as follows. Section 2 introduces the main examples of combinatorial representations of our basic commutation relations. Sections 3, 4, and 5 contain proofs of Theorems 1.1, 1.2, and 1.3, respectively. Corollary 4.2 establishes the Schur positivity of stable Grothendieck polynomials. We also explain in Sections 4 and 5 how the usual Littlewood–Richardson and Murnaghan–Nakayama rules appear as special cases of formulas (1.4) and (1.5) of Theorems 1.2 and 1.3. Corollary 5.2 contains a formula for characters χ_w associated with stable Schubert polynomials, generalizing the Murnaghan–Nakayama rule. Corollary 5.3 gives a new combinatorial formula for irreducible S_n-characters, expressed entirely in the language of the plactic monoid. In Section 6, we suggest various other examples to which the theory also applies.

We should mention the important recent paper of Gelfand et al. [13], which also develops a theory of noncommutative symmetric functions, but in a setting different from ours. On the surface, our results do not seem to overlap, and the main emphasis of our work is quite different. For example, our symmetric functions always commute, but this phenomenon does not appear in their theory. However, it is clear that deeper connections between the two approaches remain to be explored.

2. Main examples

In this section we introduce several fundamental examples of combinatorial representations of commutation relations (1.1)/(1.2) and (1.3). We also describe the corresponding families of symmetric functions $F_{h/g}$.

Example 2.1 (*The plactic algebra*; Schützenberger [25,21]). This is a quotient of the free associative algebra under the Robinson–Schensted homomorphism. Knuth [17] gave the complete list of equivalence relations which, in our current terminology, mean that the plactic algebra is defined by (1.2) together with the additional 'local Knuth relations'

$$\begin{aligned} u_{i+1}u_iu_i &= u_iu_{i+1}u_i, \\ u_{i+1}u_iu_{i+1} &= u_{i+1}u_{i+1}u_i. \end{aligned} \tag{2.1}$$

Note that (1.3) is simply the sum of these two relations. The plactic algebra is a semigroup algebra of the plactic monoid (defined by (1.2) and (2.1)) whose remarkable properties were developed in [25,21].

In this example, as well as several others, the associated symmetric functions are constructed according to the following general scheme. Let A be an algebra with identity generated by $u_1, \ldots, u_n$, all defining relations being either of the form $w = w'$ or of the form $w = 0$ where w and w' are words in the alphabet $\{u_1, \ldots, u_n\}$. Then the set $\mathcal{M}$ of non-vanishing equivalence classes of words is a linear basis of A. (The empty word corresponds to the identity of A.) The generators u_i act on $\mathcal{M}$ as partial maps defined by left (or right) multiplication. This regular representation is of course combinatorial in our sense. Let F_h denote $F_{h/g}$ when $g = 1$. Then F_h is simply the coefficient of h in the expansion of the noncommutative Cauchy product

$$\prod_{i=1}^{m} \prod_{j=n}^{1} (1 + x_i u_j). \tag{2.2}$$

In the case of the plactic algebra, F_h is just the Schur function s_{λ^*} where λ is the shape associated to a plactic class h via the Robinson–Schensted correspondence. This statement essentially amounts to the fact that s_λ is equal to the sum of fundamental quasi-symmetric functions $F_{\mathrm{des}(T)}$ (using the notation of [14]) corresponding to descent sets of all standard tableaux T of shape λ.

Example 2.2. *The nilCoxeter algebra* of the symmetric group [11] has been already introduced in Section 1. A basis of this algebra is formed by permutations in S_{n+1}, with multiplication

$$v \cdot w = \begin{cases} \text{usual product } vw & \text{if } l(vw) = l(v) + l(w), \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, this algebra can be defined by (1.1) and

$$\begin{aligned} &u_{i+1}u_iu_{i+1} = u_iu_{i+1}u_i, \\ &u_i^2 = 0. \end{aligned} \tag{2.3}$$

Obviously, (2.3) implies (1.3).

Nonvanishing nilCoxeter equivalence classes of words are naturally labeled by permutations; each such class consists of all reduced words (in the sense of Coxeter groups) of a given permutation. The functions F_h obtained by expanding the noncommutative Cauchy product (2.2) in the basis of permutations are, by definition, the stable Schubert, or Stanley polynomials [26]:

$$F_h(x_1, \ldots, x_m) = \sum_{a_1,\ldots,a_l} \sum_{\substack{b_1 \leqslant \cdots \leqslant b_l \\ a_i \leqslant a_{i+1} \Longrightarrow b_i < b_{i+1}}} x_{b_1} \cdots x_{b_l},$$

where the summation is over all reduced words $a = a_1 \ldots a_l$ for a permutation h of length l and over all sequences $b = (b_1, \ldots, b_l)$ which are 'compatible' with a in the sense indicated. See [2,11,9] for more details of this construction. (Strictly speaking, the polynomials we have defined should be written as $F_{h^{-1}}$ in the notation of [26].)

Another faithful representation of the nilCoxeter algebra is given by *divided difference operators* [1,4].

Example 2.3 (*The nilplactic algebra* [22]). This is a 'Knuth analog' of the nilCoxeter algebra of the previous example. The defining nilplactic, or Coxeter–Knuth, relations are (1.2) and (2.3). This algebra can also be defined in a way similar to the plactic case, using the Edelman–Greene correspondence (see, e.g., [5]).

As in the case of the plactic algebra, the functions F_h associated with the regular representation of the nilplactic algebra are the *Schur functions* [22,5].

Example 2.4. (*The nil-Temperley-Lieb algebra* [2,12]). This algebra can be defined as the algebra of 321-avoiding permutations, which are the permutations $a_1 \cdots a_n$ not containing a subsequence $\cdots a_i \cdots a_j \cdots a_k \cdots$ with $a_i > a_j > a_k$. The multiplication in the nil-Temperley-Lieb algebra is inherited from the nilCoxeter algebra, as follows:

$$v * w = \begin{cases} \text{nilCoxeter product } v \cdot w & \text{if } vw \text{ is 321-avoiding;} \\ 0 & \text{otherwise.} \end{cases}$$

The defining relations for this algebra are (1.1) and

$$\begin{aligned} &u_{i+1}u_iu_{i+1} = u_iu_{i+1}u_i = 0, \\ &u_i^2 = 0. \end{aligned} \tag{2.4}$$

A reformulation of a result in [2] shows that the nil-Temperley-Lieb algebra can also be defined as an algebra of operators u_i acting on Ferrers shapes according to the following rule:

$$u_i(\lambda) = \begin{cases} \lambda \cup \{\text{box in the } i\text{th diagonal}\} & \text{if this gives a valid shape;} \\ 0 & \text{otherwise.} \end{cases}$$

Here we assume that all shapes lie inside a strip of adjacent diagonals, numbered $1, 2, \ldots, n$ from bottom to top.

From the last definition, one can immediately derive that in the case of the nil-Temperley-Lieb algebra, the $F_{h/g}$ are the *skew Schur functions* (cf. [2]). Indeed, the product

$$\prod_{j=n}^{1} (1 + xu_j)$$

can be viewed as an operator that adds a horizontal strip to a given Ferrers shape in all possible ways, each time multiplying by the power of x that is determined by the length of a strip. With $h = \lambda$ and $g = \mu$, we have

$$\prod_{i=1}^{m} \prod_{j=n}^{1} (1 + x_iu_j)\ \mu \ = \ \sum_{T} x^T\ \lambda,$$

where the sum is over all (reverse) semistandard tableaux T of shape λ/μ, and therefore

$$F_{\lambda/\mu} = \sum_{T} x^T = s_{\lambda/\mu}.$$

Example 2.5. *The degenerate Hecke algebra* $H_n(0)$. This algebra (sometimes also called the 0-Hecke algebra [3]) is the semigroup algebra of the *0-Hecke monoid* defined by (1.1) and

$$\begin{aligned} &u_{i+1}u_iu_{i+1} = u_iu_{i+1}u_i, \\ &u_i^2 = u_i \end{aligned} \tag{2.5}$$

(the usual definition can be obtained by changing the signs of the generators). The elements of the 0-Hecke monoid are in a natural bijection with permutations in S_{n+1}. In our current terminology, each 0-Hecke equivalence class contains precisely one nilCoxeter class, i.e., it contains reduced words of exactly one permutation h, and may then be identified with h. The regular action of generators u_i on permutations

(= 0-Hecke classes) is given by 'unsorting operators' similar to the *sorting operators* of [10]:

$$u_i \cdot w = \begin{cases} s_i w, & \text{if } l(s_i w) = l(w) + 1 \\ w, & \text{otherwise} \end{cases}$$

The functions F_h obtained by expanding (2.2) in the basis of permutations are the *stable β-polynomials* in the terminology of [10], with $\beta = 1$. Up to a change of sign of the variables, these non-homogeneous symmetric functions coincide with the *stable Grothendieck polynomials*, obtained from the Grothendieck polynomials of Lascoux and Schützenberger [20].

It seems worthwhile to restate at this point our definition of the functions F_h in this case. Let $h \in S_{n+1}$ be a permutation of length $l = l(h)$. Then

$$F_h(x_1, \ldots, x_m) = \sum_{L \geqslant l} \sum_{a_1, \ldots, a_L} \sum_{\substack{b_1 \leqslant \cdots \leqslant b_L \\ a_i \leqslant a_{i+1} \Rightarrow b_i < b_{i+1}}} x_{b_1} \cdots x_{b_L}$$

where the summation is over all words $a = a_1 \ldots a_L$ in the 0-Hecke equivalence class of h and over all compatible sequences $b = (b_1, \ldots, b_L)$. Note that the homogeneous component of smallest degree in F_h is obtained by summing over reduced words for h, and is therefore precisely the stable Schubert polynomial corresponding to h.

Another faithful representation of $H_n(0)$ is provided by the *isobaric divided differences* [4,20].

Example 2.6 (*Schur operators* [6] and the '*local plactic algebra*'). The latter is the 'partially commutative' analog of the plactic algebra. Its defining relations are (1.1) and (2.1). The local plactic monoid has been studied recently by C. Reutenauer (private communication), who computed its noncommutative generating series. An (unfaithful) representation of this monoid is given by the *Schur operators* u_i acting on Ferrers shapes by

$$u_i(\lambda) = \begin{cases} \lambda \cup \{\text{box in the } i\text{th column}\} & \text{if this gives a valid shape;} \\ 0 & \text{otherwise.} \end{cases}$$

The complete list of relations for the algebra of Schur operators is unknown.

For the Schur operators, the $F_{h/g}$ are the *skew Schur functions* [6]. The reason for this is exactly the same as in Example 2.4.

3. Proof of Theorem 1.1

In this section we give a proof of our first main result, namely, that the functions $s_{\lambda/\mu}(\boldsymbol{u})$ multiply and expand as ordinary skew Schur functions do, assuming relations (1.2) and (1.3) are satisfied. The proof is organized as follows. The first step is to show that noncommutative analogs of the elementary symmetric functions commute, i.e.,

$$e_j(\boldsymbol{u})e_k(\boldsymbol{u}) = e_k(\boldsymbol{u})e_j(\boldsymbol{u}) \tag{3.1}$$

for all j and k. Then we show that the $s_{\lambda/\mu}(\boldsymbol{u})$ can be expressed as polynomials in the $e_k(\boldsymbol{u})$ in the standard way, using the Jacobi–Trudi determinant. This step is based on a noncommutative analog of the Gessel–Viennot path-switching argument [15] that may be used to derive the same result in the commutative case. Once (3.1) and the Jacobi–Trudi identity have been proved, Theorem 1.1 follows immediately.

The first lemma is borrowed from [7], where it appears in a somewhat more general context. We reproduce the proof here in order to make the present discussion self-contained. Note that the conditions of Lemma 3.1 are implied by (1.2) and (1.3). In fact, the statements and proofs of our main results remain valid under the weaker hypotheses of Lemma 3.1.

Lemma 3.1 ([7]). *Assume that elements $u_1,\dots,u_n$ of an associative algebra satisfy the 'strict Knuth relations'*

$$\begin{aligned} u_iu_ku_j &= u_ku_iu_j, \quad i<j<k,\\ u_ju_iu_k &= u_ju_ku_i, \quad i<j<k, \end{aligned} \tag{3.2}$$

and the relation

$$u_ju_i(u_i+u_j) = (u_i+u_j)u_ju_i, \quad i<j.$$

Then the noncommutative analogs of elementary symmetric functions

$$e_k(u_1,\dots,u_n) = \sum_{a_1>a_2>\cdots>a_k} u_{a_1}u_{a_2}\cdots u_{a_k}$$

commute.

Proof. For $i\leqslant j$, let

$$E_{ji}(x) = (1+xu_j)(1+xu_{j-1})\cdots(1+xu_i) = \sum_k x^k e_k(u_i,\dots,u_j)$$

where x commutes with all of the u_i. Also define $E_{ji}=1$ if $i=j+1$.

The statement of the lemma is equivalent to saying that $E_{n1}(x)$ and $E_{n1}(y)$ commute, x and y being scalar variables. Let us prove that $E_{ji}(x)$ and $E_{ji}(y)$ commute, using induction on $j-i$.

The cases $j-i=-1$ and $j-i=0$ are trivial. To carry out the induction step, first note that, for $i<j$,

$$E_{j-1,i+1}(x)(u_iu_j-u_ju_i)E_{j-1,i+1}(y) = u_iu_j-u_ju_i$$

because, e.g., each nonconstant term of $E_{j-1,i+1}(x)$ is canceled by the multiplication by $u_iu_j-u_ju_i$ by virtue of (3.2). Now, for $j-i\geqslant 1$, using the induction hypothesis, one obtains

$$\begin{aligned}
&E_{ji}(x)E_{ji}(y)\\
&\quad= E_{j,i+1}(x)(1+xu_i)(1+yu_j)E_{j-1,i}(y)\\
&\quad= E_{j,i+1}(x)(1+yu_j)(1+xu_i)E_{j-1,i}(y)+xyE_{j,i+1}(x)(u_iu_j-u_ju_i)E_{j-1,i}(y)\\
&\quad= E_{j,i+1}(x)E_{j,i+1}(y)\left(E_{j-1,i+1}(y)\right)^{-1}\left(E_{j-1,i+1}(x)\right)^{-1}E_{j-1,i}(x)E_{j-1,i}(y)\\
&\qquad+xy(1+xu_j)(u_iu_j-u_ju_i)(1+yu_i)\\
&\quad= E_{j,i+1}(y)E_{j,i+1}(x)\left(E_{j-1,i+1}(x)\right)^{-1}\left(E_{j-1,i+1}(y)\right)^{-1}E_{j-1,i}(y)E_{j-1,i}(x)\\
&\qquad+xy(1+xu_j)(u_iu_j-u_ju_i)(1+yu_i)\\
&\quad= E_{j,i+1}(y)(1+xu_j)(1+yu_i)E_{j-1,i}(x)+xy(1+xu_j)(u_iu_j-u_ju_i)(1+yu_i)\\
&\quad= E_{j,i+1}(y)(1+yu_i)(1+xu_j)E_{j-1,i}(x)+xyE_{j,i+1}(y)(u_ju_i-u_iu_j)E_{j-1,i}(x)\\
&\qquad+xy(1+xu_j)(u_iu_j-u_ju_i)(1+yu_i)\\
&\quad= E_{ji}(y)E_{ji}(x)+xy(1+yu_j)(u_ju_i-u_iu_j)(1+xu_i)\\
&\qquad+xy(1+xu_j)(u_iu_j-u_ju_i)(1+yu_i)\\
&\quad= E_{ji}(y)E_{ji}(x)+(x^2y-xy^2)(u_ju_i(u_i+u_j)-(u_i+u_j)u_ju_i)\\
&\quad= E_{ji}(y)E_{ji}(x)\ ,
\end{aligned}$$

as desired. □

We note that the problem of giving conditions which guarantee that the $e_k(\boldsymbol{u})$ commute was also considered by Lascoux and Schützenberger in [21, Theorem 2.16]. The focus of [21] was on semigroup algebras defined by weight-preserving congruences. (Among our examples of Section 2, only the plactic algebra and its quotient from Example 2.6 are of this kind.) In this generality, and with some additional restrictions which are different from ours, they show that any solution of (3.1) is a quotient of the plactic algebra.

Now suppose that λ/μ is a skew shape. Define a function $\mathfrak{J}_{\lambda/\mu}(\boldsymbol{u})$ in variables $u_1, u_2, \ldots, u_n$ by

$$\mathfrak{J}_{\lambda/\mu}(\boldsymbol{u}) = \det(\, e_{\lambda_i'-\mu_j'+j-i}(\boldsymbol{u})\,), \tag{3.3}$$

where by definition $e_0(\boldsymbol{u}) = 1$ and $e_k(\boldsymbol{u}) = 0$ if $k < 0$. This is the analog of the standard Jacobi–Trudi expansion for $s_{\lambda/\mu}$ in the commutative case. By Lemma 3.1 it is not necessary to specify the order in which terms in the expansion of (3.3) are multiplied, so $\mathfrak{J}_{\lambda/\mu}(\boldsymbol{u})$ is well-defined. As noted above, it will suffice to prove the following lemma.

Lemma 3.2. *Suppose that the u_i satisfy* (1.2) *and* (1.3). *Then*

$$\mathfrak{J}_{\lambda/\mu}(\boldsymbol{u}) = s_{\lambda/\mu}(\boldsymbol{u}).$$

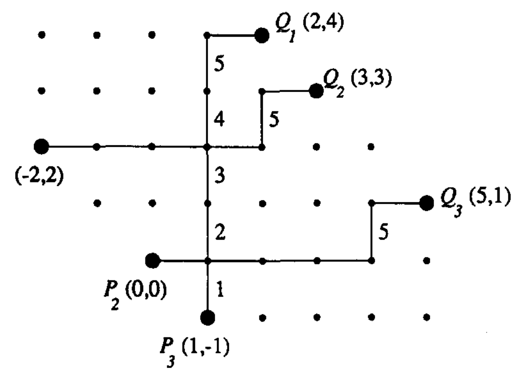

Fig. 1. $\lambda = (3,3,3,2)$, $\mu = (3,1)$, $m(\pi_{21}) = 5432$, $m(\pi_{12}) = 5$, $m(\pi_{33}) = 51$, $n = 6$.

Proof. Following [15] we may identify individual monomials appearing in the expansion of (3.3) with families of lattice paths $(\pi_{1j_1}, \pi_{2j_2}, \ldots)$, where π_{ij} denotes a path from $P_i = ((i-1)-\mu_i', -(i-1)+\mu_i')$ to $Q_j = (n+(j-1)-\lambda_j', -(j-1)+\lambda_j')$. To each path π_{ij} we associate a noncommutative monomial $m(\pi_{ij})$ obtained by taking the product of u_k with decreasing indices k corresponding to vertical segments of the path, where a vertical segment from $(x, y-1)$ to (x, y) has been labeled by $x+y$. An example is illustrated in Fig. 1. Each term appears with the sign of the permutation that matches initial points P_i with terminal points Q_j. It is easy to see that for fixed i and j the expression $e_{\lambda_i'-\mu_j'+j-i}(\boldsymbol{u})$ may be obtained by summing the monomials $m(\pi_{ij})$ over all paths π_{ij} from P_i to Q_j.

We may assume that the expansion of (3.3) is carried out by multiplying the $e_{\lambda_i'-\mu_j'+j-i}(\boldsymbol{u})$ in order of increasing i. This means that the order of multiplication of path monomials $m(\pi_{ij})$ is top-down with respect to the corresponding endpoints Q_j.

The key idea of [15] is to show that in the expansion of (3.3) all contributions obtained from families with intersecting paths cancel, and that 'non-intersecting' families correspond exactly to semi-standard skew tableaux. We will show that this argument can be carried out in the non-commutative case as well, although some additional effort and care is required.

Suppose that $(\pi_{1j_1}, \pi_{2j_2}, \ldots)$ is a family in which at least one pair of paths intersects, say, at a lattice point (α, β). Let us choose the intersection point in a canonical way, with the pair $(\alpha+\beta, \alpha-\beta)$ lexicographically maximal. The corresponding two paths must have consecutive terminal points, say Q_k and Q_{k+1}. Let $\pi_{i,k}$ denote the first path and $\pi_{j,k+1}$ the second. Then $m(\pi_{i,k})m(\pi_{j,k+1})$ may be factored as

$$m(\pi_{i,k})m(\pi_{j,k+1}) = A\, pX\, B\, Y,$$

where X, Y, A, and B are decreasing strings of indices such that $a, b > p > x, y$ for all $a \in A$, $b \in B$, $x \in X$, and $y \in Y$. Here p labels the vertical segment in $\pi_{i,k}$ that lies just above (α, β). A typical example is illustrated in Fig. 2.

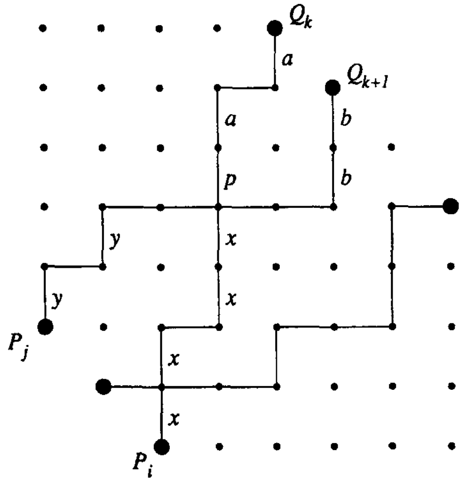

Fig. 2. $\pi_{ik} = aapxxxx$, $\pi_{j,k+1} = bbyy$.

From the fact that the paths from (α, β) to Q_k and Q_{k+1} are non-intersecting it follows that strings A and B represent the columns of a tableau, i.e., if $A = a_1 a_2 \ldots$ and $B = b_1 b_2 \ldots$, then $a_1 \leqslant b_1$, $a_2 \leqslant b_2, \ldots$

Lemma 3.3. *With notation as above, the identity*

$$A\,pX\,B\,Y \;=\; A\,pB\,X\,Y$$

can be derived using (1.2) *alone.*

Proof. We omit the details of this argument, which may be carried out by standard jeu-de-taquin transformations [25]. □

Now consider the contribution to (3.3) by all configurations which agree with our original family of paths $(\pi_{1j_1}, \pi_{2j_2}, \ldots)$ everywhere except on the initial segments of $\pi_{i,k}$ and $\pi_{j,k+1}$, i.e., the portions that lie below (α, β). The sum of all such terms may be expressed as

$$W_1\,A\;p\,e_r(u_1, \ldots, u_{\alpha+\beta})\,B\,e_s(u_1, \ldots, u_{\alpha+\beta})\,W_2,$$

where W_1 and W_2 are words obtained from paths not involving P_j, P_k, Q_k, Q_{k+1}. We can apply Lemma 3.3 to each term in the expansion of this product, transforming it into

$$W_1\,A\;p\,B\,e_r(u_1, \ldots, u_{\alpha+\beta})\,e_s(u_1, \ldots, u_{\alpha+\beta})\,W_2.$$

By Lemma 3.1, this expression is equal to

$$W_1\,A\;p\,B\,e_s(u_1, \ldots, u_{\alpha+\beta})\,e_r(u_1, \ldots, u_{\alpha+\beta})\,W_2,$$

which can in turn be rewritten (using Lemma 3.3 again) as

$$W_1 A\, p\, e_s(u_1,\dots,u_{\alpha+\beta})\, B\, e_r(u_1,\dots,u_{\alpha+\beta})\, W_2.$$

This is exactly the expression for the family of paths obtained from the original one by switching our two paths at the point (α,β). Hence, we have constructed a sign-reversing involution which leaves only terms of (3.3) corresponding to nonintersecting families of paths.

Now the argument continues exactly as in the commutative case. It is easy to see that if a family consists of non-intersecting paths π_{ii} from P_i to Q_i, the product $m(\pi_{11})m(\pi_{22})\cdots$ may be interpreted as the column word of a skew tableau, and conversely. This completes the proof of Lemma 3.2, and hence of Theorem 1.1. □

4. Noncommutative Cauchy identities and a generalized Littlewood–Richardson Rule

The next two sections are devoted to applications of our main theorem on noncommutative Schur functions to the theory of (ordinary) symmetric functions and corresponding representations of the symmetric groups.

As we have already stated, any identity of the commutative theory involving only Schur functions can be extended to the case of noncommutative variables satisfying (1.2) and (1.3). The most important cases when noncommutative versions exist and have significant applications are the classical Cauchy identities

$$\prod_i \prod_j (1+x_i y_j) = \sum_\lambda s_\lambda(\boldsymbol{x}) s_{\lambda'}(\boldsymbol{y})$$

and

$$\prod_i \prod_j (1-x_i y_j)^{-1} = \sum_\lambda s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}),$$

where $\boldsymbol{x}=(x_1,\dots,x_m)$ and $\boldsymbol{y}=(y_1,\dots,y_n)$.

To replace the y_i by noncommuting u_i, we need to choose an appropriate order of factors in the double products; this allows us to express the left-hand sides in terms of noncommutative analogs of elementary and complete homogeneous symmetric functions.

Theorem 4.1 (Noncommutative Cauchy identities). *Let u_j be elements of an associative algebra satisfying* (1.2) *and* (1.3). *Let x_i be commuting indeterminates, also commuting with each of the u_j. Then*

$$\prod_{i=1}^{m} \prod_{j=n}^{1} (1+x_i u_j) = \sum_\lambda s_\lambda(\boldsymbol{x}) s_{\lambda'}(\boldsymbol{u}) \tag{4.1}$$

and

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1-x_iu_j)^{-1}=\sum_{\lambda}s_\lambda(\boldsymbol{x})s_\lambda(\boldsymbol{u}),$$

where the factors in the double products are multiplied in the specified order.

Proof. For instance, the left-hand side of (4.1) can be rewritten as $\prod_{i=1}^{m}\sum_k x_i^k e_k(\boldsymbol{u})$, making this identity an immediate corollary of Theorem 1.1. □

Proof of Theorem 1.2 The noncommutative Cauchy identity (4.1) implies that

$$\begin{aligned}F_{h/g}(\boldsymbol{x}) &= \left\langle \sum_{\lambda} s_\lambda(\boldsymbol{x})s_{\lambda'}(\boldsymbol{u})\ g\ ,\ h \right\rangle\\ &= \sum_{\lambda}\langle s_{\lambda'}(\boldsymbol{u})g,h\rangle\, s_\lambda(\boldsymbol{x}).\end{aligned}$$

In other words,

$$c_{g\lambda}^{h}=\langle s_{\lambda'}(\boldsymbol{u})g,h\rangle\,, \tag{4.2}$$

which is equivalent to (1.4). □

Schur-positivity of ordinary skew Schur functions is well known, and the coefficients (1.4) are given by the classical Littlewood–Richardson rule. From Examples 2.4 and 2.6 one obtains proofs of two different but essentially equivalent versions of this rule (cf.[8]).

In the nilCoxeter case (Example 2.2) Schur-positivity of the functions F_h was conjectured in [26] and proved independently in [5] and [22]. The corresponding specialization of formula (1.4) is a reformulation of the combinatorial rules obtained in [5,22].

The generalization of these results to the degenerate Hecke algebra (Example 2.5) is new.

Corollary 4.2 (Schur positivity of stable Grothendieck polynomials). *Every homogeneous component of a stable β-polynomial is a nonnegative integer combination of Schur functions.*

For the original stable Grothendieck polynomials that appear as limiting cases of the Grothendieck polynomials of Lascoux and Schützenberger [20], Corollary 4.2 means that they are 'Schur-alternating', i.e., the homogeneous component of degree $l(h)+k$ is Schur-positive for k even and Schur-negative otherwise. This is because the corresponding definition of the Hecke algebra involves the relation $u_i^2=-u_i$, which introduces negative signs throughout.

5. A generalized Murnaghan–Nakayama rule

This section contains the proof of Theorem 1.3 and some applications thereof.

Proof of Theorem 1.3. Let $\chi_{h/g}$ denote the character of the symmetric group that corresponds to the function $F_{h/g}$. The value of $\chi_{h/g}$ at a conjugacy class specified by a composition $\alpha=(\alpha_1,\ldots,\alpha_k)$ is equal to

$$\chi_{h/g}(\alpha)=\sum_{\lambda} c^h_{g\lambda}\ \chi_\lambda(\alpha), \tag{5.1}$$

where χ_λ denotes an irreducible character and $c^h_{g\lambda}$, as before, is the corresponding multiplicity given by (4.2).

The irreducible characters χ_λ may be computed by the Frobenius formula

$$p_\alpha(\boldsymbol{x})=\mathrm{sgn}(\alpha)\sum_{\lambda}\chi_\lambda(\alpha)s_{\lambda'}(\boldsymbol{x}), \tag{5.2}$$

where $p_\alpha=p_{\alpha_1}\cdots p_{\alpha_k}$ is the power sum symmetric function and $\mathrm{sgn}(\alpha)$ is equal to $\prod(-1)^{\alpha_i-1}$. In order to obtain a noncommutative version of (5.2) it is necessary to express the left-hand side in terms of Schur functions. Note that simply reproducing p_α verbatim from its original definition obviously does not work. We can use instead a well-known expression for the p_j via *hook Schur functions*. Using this approach, one obtains a noncommutative analog of a power sum by defining

$$p_j(\boldsymbol{u})=\sum_{\substack{\lambda\vdash j\\ \lambda=\text{hook shape}}}(-1)^{l(\lambda)-1}s_\lambda(\boldsymbol{u})=\sum_{a=(a_1,\ldots,a_j)}(-1)^{\mathrm{des}(a)}u_{a_1}\cdots u_{a_j}, \tag{5.3}$$

where the second sum is over all hook words a of length j, and $\mathrm{des}(a)$ counts the number of strict descents in a. Then we define $p_\alpha(\boldsymbol{u})=p_{\alpha_1}(\boldsymbol{u})\cdots p_{\alpha_k}(\boldsymbol{u})$.

Theorem 1.1 allows to write down the complete noncommutative analog of the Frobenius formula (5.2), which together with (5.1) and (4.2) gives

$$\begin{aligned}\chi_{h/g}(\alpha)&=\sum_{\lambda}c^h_{g\lambda}\ \chi_\lambda(\alpha)\\ &=\sum_{\lambda}\langle s_{\lambda'}(\boldsymbol{u})g,h\rangle\,\chi_\lambda(\alpha)\\ &=\left\langle\sum_{\lambda}\chi_\lambda(\alpha)s_{\lambda'}(\boldsymbol{u})g,h\right\rangle\\ &=\mathrm{sgn}(\alpha)\,\langle p_\alpha(\boldsymbol{u})g,h\rangle,\end{aligned}$$

a formula which is equivalent to (1.5) by virtue of (5.3). □

Another proof of Theorem 1.3 could be obtained from the noncommutative version of the well-known Cauchy-type identity for the power sum symmetric functions.

In cases where the u_i satisfy the commutation relations (1.1) and (2.3) of the nil-Coxeter algebra (cf. Examples 2.2 and 2.4), the combinatorial rule of Theorem 1.3 can be further simplified. Suppose that one of the sequences w_i from (1.5) contains indices a and c but does not contain an index b such that $a < b < c$. Take such an occurrence with the smallest possible value of c. Then $c-1$ is not present in w_i and, by (1.1), c commutes with the substring (necessarily nonempty) consisting of elements strictly less than $c-1$. Exchanging c with this segment results in another hook word whose ascent number has a different parity; hence the contributions of these two words to (1.5) cancel each other. (Note that the condition $u_c^2 = 0$ guarantees that w_i only contains one copy of c.) Thus we arrive at the following modification of Theorem 1.3.

Theorem 5.1. *Assume that a combinatorial representation of the relations* (1.1) *and* (2.3) *is given. Then the character value* $\chi_{h/g}(\alpha)$ *is given by* (1.5) *where each* w_i *is a hook word of length* α_i, *and the set of entries of* w_i *is an interval of* $\mathbb{Z}$.

In the case of the nil-Temperley–Lieb algebra (Example 2.4), Theorem 5.1 gives exactly the standard Murnaghan–Nakayama rule. To see this, recall that u_i is an operator on shapes that adds a cell in the ith diagonal. It is easy to check that a hook word w adds cells in a (possibly disconnected) border strip, i.e. a collection of cells not containing a 2×2 square. This border strip is connected if and only if the set of entries of w forms an interval of $\mathbb{Z}$. Conversely, given a connected border strip, there is exactly one way to add cells so that the corresponding operators form a hook word. To illustrate, consider the following border strip, with diagonals numbered as shown:

$$\begin{array}{cccc} & \boxed{5} & \boxed{6} & \boxed{7} \\ & \boxed{4} & & \\ \boxed{2} & \boxed{3} & & \\ \boxed{1} & & & \end{array}$$

The unique hook word producing this shape is $w = 7631245$ (recall that operations are performed from right to left). The 'arm' of w contains exactly one element from each row, hence $(-1)^{\mathrm{asc}(w)}$ is exactly the weight required by the Murnaghan–Nakayama formula.

For the nilCoxeter algebra, the formula expressed by Theorem 5.1 is new.

Corollary 5.2 (Murnaghan–Nakayama formula for stable Schubert polynomials). *The character* χ_w *of the representation of the symmetric group associated with the stable Schubert polynomial for a permutation* w *(see* [18,19]) *is given by*

$$\chi_w(\alpha) = \sum_{w_1 \cdots w_k \in Red(w)} (-1)^{\mathrm{asc}(w_1)+\cdots+\mathrm{asc}(w_k)},$$

where the summation is over all hook words $w_1, \ldots, w_k$ *of lengths* $\alpha_1, \ldots, \alpha_k$, *such that the concatenation* $w_1 \ldots w_k$ *is a reduced word for* w, *and the set of entries of* w_i *forms an interval of* $\mathbb{Z}$.

Table 1

α	4123	1423	1243	$\chi_\lambda(\alpha)$
(4)	4123 (−)	...	...	−1
(31)	412 • 3 (−)	...	124 • 3 (+)	0
(22)	41 • 23 (−)	14 • 23 (+)	12 • 43 (−)	−1
(21^2)	41 • 2 • 3 (−)	14 • 2 • 3 (+)	12 • 4 • 3 (+)	+1
(1^4)	4 • 1 • 2 • 3 (+)	1 • 4 • 2 • 3 • (+)	1 • 2 • 4 • 3 (+)	+3

A corresponding result for stable Grothendieck polynomials can be obtained by specializing Theorem 1.3 to the case of the degenerate Hecke algebra.

We conclude this section by giving a new formula for irreducible S_n-characters, obtained by interpreting Theorem 1.3 in the plactic monoid algebra.

Corollary 5.3 (Plactic version of the Murnaghan–Nakayama rule). *If χ_λ is an irreducible character of the symmetric group, then the value of χ_λ on a conjugacy class $\alpha = (\alpha_1, \ldots, \alpha_k)$ is given by*

$$\chi_\lambda(\alpha) = \sum_{P_1 \ldots P_k = P} (-1)^{\mathrm{des}(P_1)+\cdots+\mathrm{des}(P_k)},$$

where P is a fixed element of the plactic monoid corresponding to a tableau word of shape λ, and the sum is over all factorizations of P into hook words $P_1, \ldots, P_k$ of lengths $\alpha_1, \ldots, \alpha_k$.

The conversion from ascents to descents in the statement of Corollary 5.3 is necessary because in the plactic algebra we have $F_h = s_{\lambda^*}$, where λ is the shape associated to h (see Example 2.1). Thus, a direct application of Theorem 1.3 computes characters corresponding to λ^* rather than λ, and Corollary 5.3 is obtained by writing all words in reverse order.

As an illustration of Corollary 5.3, we compute the character values $\chi_\lambda(\alpha)$, where $N = 4$ and $\lambda = (3, 1)$. We fix a plactic class $P = \{4123, 1423, 1243\}$ corresponding to shape λ (under the Robinson–Schensted map). Table 1 shows all factorizations of elements of P into hook words, arranged by type, and indicating the weight ($+1$ or -1) associated with each factorization.

The original Murnaghan–Nakayama formula can be derived from Corollary 5.3, by an argument similar to the proof of Theorem 5.1. Using well-known properties of the Robinson–Schensted correspondence, it is easy to show that inserting a hook word into a tableau of shape μ yields a tableau whose shape differs from μ by a (possibly disconnected) border strip. If the border strip is connected, the original hook word can be uniquely recovered from the resulting tableau; if it is disconnected, there is a sign-reversing involution that cancels its contribution with that of another hook word yielding the same tableau. We leave the remaining details of this argument to the reader.

6. More examples of combinatorial representations

We discuss below some general procedures which can be used to construct other representations of our basic commutation relations.

Associative algebras arising in all of our examples are, in the terminology of Vershik [27], *local stationary algebras* which means that their defining relations are (1.1) plus some commutation relations that only involve adjacent generators and are invariant under the shift maps $u_i \mapsto u_{i\pm 1}$. Many interesting examples of such algebras arise in the following well-known way.

Let V be a vector space and u a linear operator in $V \otimes V$. Now let u_i act in $V^{\otimes(n+1)}$ by $u_i = I^{(i-1)} \otimes u \otimes I^{(n-i)}$ where I is an identity operator on V. In other words, u_i acts as u in the tensor product of the ith and $(i+1)$st components. Then the locality condition (1.1) is automatically satisfied, and we only need to worry about (1.3).

Assume $\dim V = m$. Then $V^{\otimes(n+1)}$ may be viewed as the vector space formally spanned by strings of length $n+1$ whose entries belong to an alphabet $\mathcal{M}$ of size m. The operator u_i maps a string $d = d_1 \dots d_{n+1}$ to a linear combination of strings which agree with d everywhere except possibly the ith and $(i+1)$st positions. Such an action is combinatorial if it is defined by a partial map $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$. In other words,

$$u_i(\dots, d_i, d_{i+1}, \dots) = \begin{cases} (\dots, \phi(d_i, d_{i+1}), \dots) & \text{if } \phi(d_i, d_{i+1}) \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

The next several examples are of this type.

Example 6.1 (*Sweeping operators*). Suppose $\mathcal{M}$ is a monoid and ε is a *right* unit of $\mathcal{M}$. Define

$$\Phi(\alpha, \beta) = (\varepsilon, \alpha\beta)$$

for all $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$. Then the corresponding operators u_i satisfy the 'weak 0-Hecke relations'

$$\begin{aligned} u_{i+1} u_i u_{i+1} &= u_i u_{i+1} u_i \\ u_{i+1} u_i u_i &= u_{i+1} u_{i+1} u_i. \end{aligned} \tag{6.1}$$

Note that, in general, the stronger relations (2.5) of the degenerate Hecke algebra are not satisfied, since the u_i need not be idempotent.

If ε is a *left* unit of $\mathcal{M}$, then the corresponding u_i are idempotent, and the four terms $u_{i+1}u_iu_{i+1}$, $u_iu_{i+1}u_i$, and $u_{i+1}u_iu_i$, and $u_{i+1}u_{i+1}u_i$ occurring in (1.3) are equal (in fact, they are all equal to $u_{i+1}u_i$). Hence, this action simultaneously represents both the plactic relations (2.1) and the 0-Hecke relations (2.5).

Example 6.2 (*Right propagation*). Suppose $\mathcal{M}$ is a monoid in which every element is idempotent. Define

$$\Phi(\alpha, \beta) = (\alpha, \alpha\beta)$$

for all $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$. Then the corresponding operators u_i are idempotent, and satisfy (2.1) and (2.5) as in the previous example. An interesting special case occurs when multiplication in $\mathcal{M}$ is defined to be $\alpha\beta = \alpha$, for all $\alpha \in \mathcal{M}$. Then the operation Φ becomes

$$\Phi(\alpha, \beta) = (\alpha, \alpha).$$

So the action of u_i simply propagates the ith entry to the right.

Example 6.3 (*Contiguous cell growth*). Let $\mathcal{M} = \{0, 1\}$, and let Φ be the partial function defined by

$$\Phi(\alpha, \beta) = \begin{cases} (1,1) & \text{if } (\alpha, \beta) = (0,1) \text{or } (1,0), \\ 0 & \text{otherwise.} \end{cases}$$

Then the corresponding operators u_i satisfy the nil-Temperley-Lieb relations (2.4).

Example 6.4 (*A one-dimensional particle system*). Let $\mathcal{M} = \{0, 1\}$ as in the previous example, and let

$$\Phi(\alpha, \beta) = \begin{cases} (1,0) & \text{if } (\alpha, \beta) = (0,1), \\ 0 & \text{otherwise} \end{cases}$$

Then the corresponding operators u_i satisfy (2.4). In fact, this is an equivalent description of the representation of the nil-Temperley-Lieb algebra by operators acting on Ferrers shapes, discussed in Example 2.4.

Example 6.5 (*Sorting in distributive lattices*). Let $\mathcal{M}$ be a distributive lattice. Define

$$\Phi(\alpha, \beta) = (\alpha \wedge \beta, \alpha \vee \beta)$$

for all $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$. Then the generators u_i satisfy the degenerate Hecke relations (2.5). When $\mathcal{M}$ is a chain, the operation Φ becomes

$$\Phi(\alpha, \beta) = (\min\{\alpha, \beta\}, \max\{\alpha, \beta\})$$

and we get the ordinary sorting operators from [10]

Every finite distributive lattice may be represented as a sublattice of a product of chains; hence in this case the representations defined in Example 6.5 also appear as constituents of tensor products of representations based on ordinary sorting operators. One can form tensor product of other combinatorial representations and obtain new families of examples in a similar way.

If $\mathcal{M}$ is a lattice, then distributivity is essential in order for (2.5) (or, equivalently, (1.3)) to be satisfied. The following lemma gives a slightly weaker condition characterizing combinatorial actions satisfying (2.5) and constructed from (nonpartial) maps.

Lemma 6.6. *Let $\wedge$ and $\vee$ be binary operations on $\mathcal{M}$, defining a map*

$$\Phi(\alpha, \beta) = (\alpha \wedge \beta, \alpha \vee \beta)$$

from $\mathcal{M} \times \mathcal{M}$ to $\mathcal{M} \times \mathcal{M}$. In order for Φ to define an action satisfying (2.5), it is necessary and sufficient that the operations $\wedge$ and $\vee$ satisfy

$$(\alpha \wedge \beta) \wedge (\alpha \vee \beta) = \alpha \wedge \beta,$$
$$(\alpha \wedge \beta) \vee (\alpha \vee \beta) = \alpha \vee \beta,$$
$$(\alpha \wedge \beta) \vee ((\alpha \vee \beta) \wedge \gamma) = (\alpha \vee (\beta \wedge \gamma)) \wedge (\beta \vee \gamma)$$

for all α, β, and $\gamma \in \mathcal{M}$. These conditions clearly hold for distributive lattices.

The representation in the next example is noncombinatorial; its representing matrices are stochastic (Markovian).

Example 6.7 (*Bernoulli propagation*). Let $p \in (0,1)$. Define $u : V \otimes V \to V \otimes V$ by

$$u(\alpha \otimes \beta) = p \cdot \alpha \otimes \beta + (1-p) \cdot \alpha \otimes \alpha \ .$$

Then the action of u_i on a string $d = (\ldots, d_i, d_{i+1}, \ldots)$ may be described as follows: if $d_i \neq d_{i+1}$, then set $d_{i+1} \leftarrow d_i$ with probability $1-p$; otherwise leave d fixed. In general, these u_i satisfy identity (1.3) but neither of its "bijectivizations", i.e., neither the plactic relations (2.1) nor the relations (6.1).

The preceding example may be constructed from a corresponding deterministic model (Example 6.2), using the following simple but useful observation.

Lemma 6.8. *Suppose that A is an algebra with unit I and generators u_i satisfying* (1.3)*. Let p and $q \neq 0$ be fixed scalars. Define elements $\tilde{u}_i \in A$ by*

$$\tilde{u}_i = pI + qu_i.$$

Then the $\tilde{u}_i$ generate A and satisfy (1.3) (*although the stronger relations* (2.1), (2.4), (2.5), (6.1), *may not be preserved*).

Another source of interesting combinatorial representations of (1.1) is provided by actions of associative algebras on paths in graded graphs (see, e.g., [16,27]. A typical example is Young's seminormal form of an irreducible representation of (the group algebra of) the symmetric group and its q-analog for Hecke algebras. One of the simplest examples of this kind satisfying (1.3) is the representation of the nil-Temperley-Lieb algebra described in Example 6.4, which may be interpreted as a path algebra in the two-dimensional Pascal graph $\mathbb{Z}_+^2$.

For each of the examples in this section we can construct a family of characters of the symmetric (or general linear) group related to the Schur-positive functions $F_{h/g}$. The intrinsic descriptions of the corresponding representations remain to be found.

References

[1] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Schubert cells and cohomology of the spaces G/P, Russian Math. Surveys 28 (1973) 1–26.

[2] S.C. Billey, W. Jockusch, R.P. Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combin. 2 (1993) 345–374.

[3] R. Carter, Representation theory of the 0-Hecke algebra, J. Algebra 104 (1986) 89–103.

[4] M. Demazure, Desingularisation des varietés Schubert generalisées, Ann. Sc. E. N.S. (4) 7 (1974) 53–88.

[5] P. Edelman, C. Greene, Balanced tableaux, Advances in Math. 63 (1987) 42–99.

[6] S. Fomin, Schur operators and Knuth correspondences, J. Combin. Theory 72 (1995) 277–292.

[7] S. Fomin, Symmetric functions arising from associative algebras, and a condition of Malcev, in preparation.

[8] S. Fomin, C. Greene, A Littlewood–Richardson Miscellany, European J. Combin. 14 (1993).

[9] S. Fomin, A.N. Kirillov, The Yang–Baxter equation, symmetric functions, and Schubert polynomials, Discrete Math. 153 (1996) 123–143.

[10] S. Fomin, A.N. Kirillov, Grothendieck polynomials and the Yang–Baxter equation, 6th Internal. Conf. on Formal Power Series and Algebraic Combinatorics, DIMACS, 1994, pp. 183-190.

[11] S. Fomin, R.P. Stanley, Schubert polynomials and the nilCoxeter algebra, Adv. in Math. 103 (1994) 196–207.

[12] S. Fomin, X.G. Viennot, Combinatorial applications of the nil-Temperley-Lieb algebra, in preparation.

[13] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, J.-I. Thibon, Noncommutative symmetric functions, Adv. in Math. 13 (1995) 218–348.

[14] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, Contemp. Math. 34 (1984) 289–301.

[15] I. Gessel, G.X. Viennot, Binomial determinants, paths, and hook length formulae, Adv. in Math. 58 (1985) 300–321.

[16] F.M. Goodman, P. de la Harpe, V.F.R. Jones, Coxeter Graphs and Towers of Algebras, Springer–Verlag, Berlin, 1989.

[17] D.E. Knuth, Permutations, matrices, and generalized Young tableaux, Pacific J. Math. 34 (1970) 709–727.

[18] W. Kraśkiewicz, Reduced decompositions in Weyl groups, European. J. Combin. 16 (1995) 293–313.

[19] W. Kraśkiewicz, P. Pragacz, Foncteurs de Schubert, C.R. Aadc. Sci. 304 (1987) 209–211.

[20] A. Lascoux, Anneau de Grothendieck de la variete de drapeaux, in: The Grothendieck Festschrift, P. Cartier et al., Eds. vol. III, Birkhauser, Basel, 1990, pp. 1–34.

[21] A. Lascoux, M.-P. Schützenberger, Le monoide plaxique, Quad. de La ricerca scientifica 109 (1981) 129–156.

[22] A. Lascoux, M.-P. Schützenberger, Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variete de drapeau, C.R. Acad. Sci. Paris Ser. I Math. 295 (1982) 629–633.

[23] A. Lascoux, M.-P. Schützenberger, Symmetry and Flag manifolds, in: Invariant Theory, Lecture Notes in Math. 996, Springer, Berlin, 1983 pp. 118–144.

[24] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, Oxford, 1995.

[25] M.-P. Schützenberger, La correspondence de Robinson, in: D. Foata (Ed.), Combinatoire et représentation du groupe symétrique, Lecture Notes in Math. 579 (1977), springer, Berlin, 59–135.

[26] R.P. Stanley, On the number of reduced decompositions of elements of Coxeter groups, European J. Combin. 5 (1984) 359–372.

[27] A.M. Vershik, Local stationary algebras, Amer. Math. Soc. Transl. ser. 2 148 (1991) 1–13.